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BOUNDARY-VALUE PROBLEM FOR A DEGENERATE SYSTEM OF PARABOLIC EQUATIONS IN BOUNDARY-LAYER THEORY

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A problem is considered for the system describing gas flows with plate boundary layer separation in Mises variables in boundary-layer theory. The existence of generalized solutions of the problem is proved.

Key words: *boundary layer, degenerate parabolic equations, steady-state gas flow.*

1. Formulation of the Problem. In a domain $E = \{t, x\colon\ 0 < t \leqslant T,\ 0 < x < \infty\}$ we consider the first boundary-value problem for the system of equations

$$L(u, w) \cdot \boldsymbol{w} = (a(u, w)u\boldsymbol{w}_x)_x + b(t)\boldsymbol{w}_x - \boldsymbol{w}_t = 0 \qquad (a \geqslant 0) \tag{1}$$

subject to the conditions

$$\boldsymbol{w}(0, x) = \boldsymbol{m}_0(x), \qquad \boldsymbol{w}(t, 0) = \boldsymbol{m}_1, \qquad \lim_{x\to\infty} \boldsymbol{w}(t, x) = \boldsymbol{m}_\infty, \tag{2}$$

where

$$\boldsymbol{w}(t, x) = \{u(t, x), w(t, x)\}, \qquad \boldsymbol{m}_0(x) = \{u_0(x), m_0(x)\},$$

$$\boldsymbol{m}_1 = \{0, m_1\}, \qquad \boldsymbol{m}_\infty = \{u_\infty, m_\infty\}, \qquad (m_1, m_\infty, u_\infty) = \text{const} > 0.$$

In boundary-layer theory, problem (1), (2) in Mises variables describes steady-state gas flow near a plate at a Prandtl number $\text{Pr} = 1$ with gas injection [if $b(t) < 0$] or suction [if $b(t) > 0$] through the plate. In this case, $u(t, x)$ is the horizontal velocity component, $w(t, x) = u^2/2 + \theta$ is the total energy, $\theta(t, x)$ is the enthalpy, and $a(u, w) \equiv a(\theta)$ is the dynamic viscosity. In [1], problem (1), (2) was studied under the assumption that the functions $b(t)$ are nonnegative. It was proved that problem (1), (2) has a classical solution $\boldsymbol{w}(t, x)$ provided that $u(t, x) > 0$ for $(t, x) \in E$, which corresponds to a continuous gas flow over the plate.

In the present work for an arbitrary sign of the function $b(t)$, it is proved that problem (1), (2) has generalized solutions which, in particular, correspond to gas flows with plate boundary layer separation, i.e., with the formation of stagnation domains [$u(t, x) \equiv 0$]. In this case, the classical solution may not exist since the derivative of u_x becomes unbounded at a certain point [2].

We consider the class $H(E)$ of functions $\boldsymbol{q} = \{p(t, x), q(t, x)\}$ which are continuous in E, have a generalized derivative $(p\,\boldsymbol{q})_x$, and satisfy the inequalities

$$p(t, x) > 0, \qquad |q| \leqslant Mp, \qquad |(p\,\boldsymbol{q})_x| \leqslant M, \qquad (t, x) \in E,$$

where M is a certain positive constant.

Definition 1. A function $\boldsymbol{w}(t, x) = \{u(t, x), w(t, x)\}$ will be called a generalized solution of the boundary-value problem (1), (2) if $\boldsymbol{\omega} = \{u, w - m_1\} \in H(E)$, $m_1 = \text{const} > 0$ and for each function $\overline{E}$ which is finite in $f(t, x) \in C^1(\overline{E})$ and equal to zero for $x = 0$ and T, the following integral identity is satisfied:

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$$\iint\limits_E (f_t \boldsymbol{\omega} - au\boldsymbol{\omega}_x f_x - b\boldsymbol{\omega} f_x)\,dt\,dx + \int\limits_0^\infty (\boldsymbol{m}_0 - \boldsymbol{m}_1) f(0,x)\,dx = 0. \tag{3}$$

Here

$$u\boldsymbol{\omega}_x = (u\boldsymbol{\omega})_x - \frac{\boldsymbol{\omega}}{2u}(u^2)_x.$$

Let the following conditions be satisfied:

1) the function $a(u,w)$ be determined for values $u \geqslant 0$ and $w > 0$, is positive for these values and satisfies the smoothness condition

$$a(u,w) \in C^2(\Omega_N) \quad \forall\, N > 0, \qquad \Omega_N = \{u, w\colon 0 \leqslant u \leqslant 1/N,\ 1/N \leqslant w \leqslant N\};$$

2) the functions $\boldsymbol{m}_0(x) \in \overline{C}^1(0,\infty)$, $u_0^2(x) \in C^1[0,\infty)$, $b(t) \in C^\alpha[0,T]\ \forall T > 0$, $\alpha \in (0,1)$, and $\overline{C}^1(0,\infty) = C[0,\infty) \cap C^1(0,\infty)$;

3) the functions $\{u_0, u_0', m_0 - m_1, M_0 u_0' - |m_0'|\} > 0$ at $x > 0$, $u_0'(0) > 0$, and $\boldsymbol{m}_0(0) = \boldsymbol{m}_1$, $\lim\limits_{x\to\infty} \boldsymbol{m}_0(x) = \boldsymbol{m}_\infty$ (M_0 is a certain positive constant).

Theorem 1. *If conditions* 1–3 *are satisfied, the boundary-value problem* (1), (2) *has a generalized solution* $\boldsymbol{w}(t,x)$ *in the domain* E $\forall T > 0$, *and in the domain* $E^+ = \{t, x\colon u(t,x) > 0,\ (t,x) \in E\}$, *the solution* $w(t,x)$ *belongs to* $C^{2+\alpha}(E^+)$.

2. Construction of Approximate Solutions. Let $\boldsymbol{f}_n(t,x) = \{f_n^1(t,x), f_n^2(t,x)\} \in C^3(\overline{E}_n)$, $\overline{E}_n = \{t, x\colon 0 < t \leqslant T,\ 0 < x < n\}$, $\boldsymbol{f}_n(t,0) = \{1/n, m_1\}$, $\boldsymbol{f}_n(t,n) = \boldsymbol{m}_{0n}(n)$, and $\boldsymbol{m}_{0n}(x) = \{u_{0n}(x), m_{0n}(x)\} = \boldsymbol{f}_n(0,x)$.

The functions $\boldsymbol{m}_{0n}(x) \in C^3[0,\infty)$, which approximate the initial profiles $\boldsymbol{m}_0(x) = \{u_0(x), m_0(x)\}$ in the norm $\overline{C}^1(0,\infty)$, satisfy the compatibility conditions at the points $(0,0)$, $(0,n)$ of the zeroth and first orders:

$$\boldsymbol{m}_{0n} = \{1/n, m_1\}, \quad L(u_{0n}(0), m_{0n}(0)) \cdot \boldsymbol{m}_{0n}(0) = 0, \quad L(u_{0n}(n), m_{0n}(n)) \cdot \boldsymbol{m}_{0n}(n) = 0.$$

In addition, these functions satisfy the following condition:

4) $1/n < u_{0n}(x) < M_1$, $m_1 < w_{0n}(x) < M_1$, $u_{0n}'(x) > 0$, $M_1 u_{0n}' - |m_{0n}'| > 0$ at $x \in [0,n]$, $n \geqslant N > 0$, where $M_1 = \sup\limits_{x\in[0,\infty)} \{u_0, m_0, M_0\} + 1/N$ (N is a relatively large positive number).

We change the coefficient $a(u,w)u$ in system (1), assuming that $\tilde a(u,w) \equiv \chi_1(u) a(\chi_1(u), \chi_2(w))$, where $\{\chi_1(u), \chi_2(w)\} \in C^2(-\infty,\infty)$ are used to denote the cuts of the functions u and w:

$$\chi_1(u) = \begin{cases} 1/(2n), & u \leqslant 1/(2n), \\ u, & u \in [1/n, M_1], \\ 2M_1, & u \geqslant 2M_1, \end{cases} \qquad \chi_2(w) = \begin{cases} m_1/2, & w \leqslant m_1/2, \\ w, & w \in [m_1, M_1], \\ 2M_1, & w \geqslant 2M_1. \end{cases}$$

For the boundary-value problems

$$(\tilde a(u,w)\boldsymbol{w}_x)_x + b\boldsymbol{w}_x - \boldsymbol{w}_t = 0, \qquad (t,x) \in E_n; \tag{4}$$

$$\boldsymbol{w}(t,x) = \boldsymbol{f}_n(t,x) \qquad \text{at} \quad (t,x) \in \partial E_n = \overline{E}_n \setminus E_n \tag{5}$$

for $n \geqslant N$, by virtue of the presumed smoothness of the functions $\boldsymbol{f}_n(t,x)$, $\tilde a(u,w)$, and $b(t)$, solutions $\boldsymbol{w}_n(t,x)$ exist, are unique, and belong to the space $C(\overline{E}_n) \cap C^{2+\alpha}(\overline{E}_n)$ (see [3, Theorem 7.1]).

For $(t,x) \in E_n$, condition 4 allows one, using the maximum principle, to obtain the estimates

$$1/n \leqslant u_n(t,x) \leqslant M_1, \qquad m_1 \leqslant w_n(t,x) \leqslant M_1, \tag{6}$$

by virtue of which

$$\chi_1(u_n(t,x)) \equiv u_n(t,x), \qquad \chi_2(w_n(t,x)) \equiv w_n(t,x),$$

i.e., the solutions $\boldsymbol{w}_n(t,x)$ of the boundary-value problems (4) and (5) also satisfy the boundary-value problems (1) and (5).

3. Estimates of Solutions of Auxiliary Boundary-Value Problems. We prove the following lemma:

Lemma 1. *In a domain* E_n ($n > N$), *the following estimates are valid:*

$$|w_n(t,x) - w_n(t,l)| \leqslant M_2 |u_n(t,x) - u_n(t,l)|, \qquad l = 0, n; \tag{7}$$

$$u_{nx}(t,x) \geqslant 0, \qquad |w_{nx}(t,x)| \leqslant M_2\, u_{nx}(t,x); \tag{8}$$

$$|u_n \cdot \boldsymbol{w}_{nx}(t,x)| \leqslant M_2, \tag{9}$$

where M_2 is a constant independent of n.

Inequalities (7) and (8) are proved in [1] (Lemma 1) for $a(u,w) \equiv a(\theta)$, $b(t) \geqslant 0$. The proofs of estimates (7) and (8) in [1] are also valid for the case considered.

To prove (9), we multiply by $2u_n$ both sides of Eq. (1) for the component $u = u_n$ and denote

$$a_n = a(u_n, w_n), \qquad c_n = u_n \frac{\partial a_n}{\partial u_n} u_{nx} + u_n \frac{\partial a_n}{\partial w_n} w_{nx} + b.$$

For $\sigma = u_n^2$, we obtain the equation

$$\sigma_t = \sqrt{\sigma}\, a_n \sigma_{xx} + c_n \sigma_x. \tag{10}$$

Obviously, the coefficients of this equation satisfy the conditions of Theorem 1 in [4], which implies uniform boundedness in n for the derivative $\sigma_x = 2u_n u_{nx}$ and, in view of (8), for the derivative $u_n w_{nx}$ are which proves the validity of (9).

Lemma 2. *In the domain E_n, the following estimates hold:*

$$|\sigma(t_1,x) - \sigma(t_2,x)| \leqslant M_3 |t_1 - t_2|^{1/4}; \tag{11}$$

$$|w_n(t_1,x_1) - w_n(t_2,x_2)| \leqslant M_3 [\ln(|t_1-t_2| + |x_1-x_2|)]^{-1}, \tag{12}$$

where M_3 does not depend on n.

Because the derivative $\sigma_x(t,x)$ is bounded uniformly in n, it follows that, to obtain estimate (11), it is sufficient to apply Theorem 1 in [5] to the solution $\sigma(t,x)$ of Eq. (10).

To prove (12), we make a change of the required function in the equation for $w = w_n$:

$$H(t,x) = \int_0^{\omega_n} e^{-1/\xi}\, d\xi, \qquad \omega_n = w_n - m_1 \quad (m_1 = \text{const} > 0).$$

As a result, we obtain the equation

$$H_t = a_n u_n H_{xx} + bH_x + F(t,x), \tag{13}$$

where

$$F(t,x) = (a_n u_n)_x \omega_x\, e^{-1/\omega} - a_n u_n \omega_x\, e^{-1/\omega} / \omega^2 \qquad (\omega \equiv \omega_n).$$

From the estimate of the functions $F(t,x)$ and $H_x = \omega_x\, e^{-1/\omega}$, in view of inequality (7) and the inequality

$$e^{-1/\omega} \leqslant e^{-1/(M_2(u_n - 1/n))}$$

it follows that $F(t,x)$ and the derivative H_x are uniformly bounded in n. Thus, Theorem 1 in [5] applies to the solution $H(t,x)$ of Eqs. (13), whence it follows that $H(t,x)$ satisfies (in t) the Hölder condition with the Hölder constant and the exponent independent of n:

$$|H(t_1,x_1) - H(t_2,x_2)| \leqslant c_0 (|t_1-t_2| + |x_1-x_2|)^{\varkappa}, \qquad \varkappa \in (0,1). \tag{14}$$

We estimate $\delta H = H(t_1,x_1) - H(t_2,x_2)$ in terms of $\delta w = w_1 - w_2 \equiv w_n(t_1,x_1) - w_n(t_2,x_2)$. We set $\xi = \delta w\, \tau + w_2 - m_1$. Then,

$$\delta H = \int_{w_2-m_1}^{w_1-m_1} e^{-1/\xi}\, d\xi = \delta w \int_0^1 e^{-1/(\delta w\, \tau + w_2 - m_1)}\, d\tau > c_1\, e^{-c/(\delta w)}.$$

From this it follows that

$$|\ln(\delta H / c_1)| \leqslant c/(\delta w).$$

In view of (14), the last inequality is easily transformed to inequality (12). Inequalities (11) and (12) ensure uniform continuity of the sequence $\{\boldsymbol{w}_n(t,x)\}$ on each compact $E' \subset \overline{E}$.

Lemma 3. *In a domain* $\Pi = \{t, x\colon 0 < t \leqslant T,\ \beta t < x < n\}$ *at* $\beta \geqslant |b|$ *and fairly small values of* γ, $1/\lambda$, *and* $1/N$ *(independent of* $n > N$*), the function* $g(t,x) = \gamma(1 - \mathrm{e}^{-x+\beta t})\,\mathrm{e}^{-\lambda t} + 1/n$ *is the lower boundary for* $u_n(t,x)$:

$$u_n(t,x) \geqslant 1/n + \gamma(1 - \mathrm{e}^{-x+\beta t})\,\mathrm{e}^{-\lambda t}\,. \tag{15}$$

Proof. We set

$$L_0(u_n)u \equiv a_n u u_{xx} + d_n(u)u_x^2 + bu_x - u_t,$$

where

$$d_n(u) = a_n - \frac{\partial a_n}{\partial u_n}\,u - M_2 u\,\frac{\partial a_n}{\partial w_n}, \qquad a_n = a(u_n, w_n).$$

Since $|w_{nx}| \leqslant M_2 u_{nx}$, we have $L_0(u_n)u_n \leqslant L(u_n, w_n)u_n$ for $(t,x) \in E_n$. We choose the numbers γ and $1/N$ small enough so that the difference $u_n(t,x) - g(t,x) \equiv z(t,x)$ on the boundary $\partial\Pi = \overline{\Pi}\backslash\Pi$ of the domain Π is nonnegative, and we choose λ large enough so that the expression $L_0(u_n)g$ is positive. This is possible for $\beta \geqslant |b|$. Then, the following inequalities hold:

$$z(t,x)\Big|_{\partial\Pi} \geqslant 0, \qquad L_0(u_n)u_n - L_0(u_n)g = L_1(u_n)z < 0, \qquad (t,x) \in \Pi$$

$[L_1(u_n)$ is a parabolic operator$]$. By virtue of the maximum principle, these inequalities ensure that z is nonnegative everywhere in the domain of Π, and, hence, estimate (15) is valid.

REMARK 1. According to Lemma 3, the set $\Pi_{1/n}$ of points $(t,x) \in E_n$ at which $u_n(t,x) = 1/n$ belongs to the domain $E_n \setminus \Pi$.

Lemma 4. *The integrals* $\iint\limits_{E'} u_n|\boldsymbol{w}_{nx}|^2\,dx\,dt$ *over any finite domain* $E' \subset \overline{E}$ *are uniformly bounded in* n:

$$\iint\limits_{E'} u_n|\boldsymbol{w}_{nx}|^2\,dx\,dt \leqslant M_4. \tag{16}$$

Proof. We substitute $\boldsymbol{w}_n$ into (1). Multiplying each of Eqs. (1) by the corresponding component u_n or w_n, we integrate the resulting equations over the domain E'. The integrals containing the derivatives $(a_n u_n \boldsymbol{w}_{nx})_x$ are taken by parts, and the remaining integrals are estimated in modulus using (7)–(9). Simple calculations yield estimate (16).

4. Passage to the Limit as $\boldsymbol{n \to \infty}$**.** By virtue of the estimates obtained in Sec. 3, the functions $\boldsymbol{\omega}_n = \boldsymbol{w}_n - \boldsymbol{m}_1$ are uniformly continuous on each finite compact $E' \subset \overline{E}$ and the derivatives $(u_n\boldsymbol{\omega}_n)_x$ are uniformly bounded. Using the compactness principle and applying the diagonal process, we find a subsequence $\boldsymbol{\omega}_{n_k}$ such that, as $n_k \to \infty$, it converges uniformly to a certain function $\boldsymbol{\omega} = \{u(t,x), w(t,x) - m_1\}$ which has a bounded generalized derivative $(u\boldsymbol{\omega})_x$ [6, p. 42], and $(u_{n_k}\boldsymbol{\omega}_{n_k})_x$ converges to $(u\boldsymbol{\omega})_x$ weakly in $L_2(E')$. We redenote $\boldsymbol{\omega}_{n_k} \equiv \boldsymbol{\omega}_n$.

By passing to the limit in inequalities (6), (11), and (12) for $\boldsymbol{w}_n(t,x)$ as $n \to \infty$, one cane see that $u(t,x)$ and $w(t,x)$ are nonnegative and continuous in $\overline{E}$. We prove that $\boldsymbol{\omega}(t,x)$ satisfies the integral identity (3). It is clear that the integral identity (3) is satisfied by the functions $(E = E_n)$

$$\boldsymbol{\omega}_n(t,x) = \{u_n(t,x), w_n(t,x) - m_1\}.$$

The uniform convergence of $\boldsymbol{\omega}_n \to \boldsymbol{\omega}$ allows passage to the limit as $n \to \infty$ in identity (3), in particular, in the integral $\iint\limits_{E_n} a_n u_n f_x \boldsymbol{\omega}_{nx}\,dt\,dx$. We set $\varphi(t,x) = a\sqrt{u}$ and $\boldsymbol{\psi}(t,x) = \sqrt{u}\,\boldsymbol{\omega}_x$. Using $\varphi_n(t,x)$ and $\boldsymbol{\psi}_n(t,x)$ to denote the same functions for $u_n = u_n(t,x)$ and $\boldsymbol{\omega}_n = \boldsymbol{\omega}_n(t,x)$, we consider the differences

$$(\varphi_n, \boldsymbol{\psi}_n f_x) - (\varphi, \boldsymbol{\psi} f_x) = (\varphi_n - \varphi, \boldsymbol{\psi}_n f_x) + (f_x\varphi, \boldsymbol{\psi}_n - \boldsymbol{\psi})_{C\Pi_\varepsilon} + (f_x\varphi, \boldsymbol{\psi}_n - \boldsymbol{\psi})_{\Pi_\varepsilon}, \tag{17}$$

where $(f,g) = \iint\limits_{E_n} fg\,dt\,dx$, $\Pi_\varepsilon = \{t, x\colon |u_n(t,x)| \leqslant \varepsilon,\ (t,x) \in E_n\}$, and $C\Pi_\varepsilon = \overline{E}_n \setminus \Pi_\varepsilon$.

On the right side of (17), the first term tends to zero as $n \to \infty$ because of the uniform convergence φ_n to φ in $E' \subset E$. The integral over Π_ε is uniformly small with respect to n for small ε. For fixed ε, the integral over $C\Pi_\varepsilon$ tends to zero since, in this domain, the derivatives $\boldsymbol{\omega}_{nx}$ satisfy the Hölder condition uniformly in n (see [5, Theorem 4.6]) and, hence, in the domain $C\Pi_\varepsilon$, $\boldsymbol{\omega}_{nx} \to \boldsymbol{\omega}_x$. By virtue of the arbitrariness of ε, from this we obtain $\lim(\varphi_n, \boldsymbol{\psi}_n f_x) = (\varphi, \boldsymbol{\psi} f_x)$.

Thus, the limiting function $\boldsymbol{\omega} \equiv \boldsymbol{w} - \boldsymbol{m}_1$ belongs to the class $H(E)$ and satisfies the integral identity (3). We prove that, in the domain where $u(t,x) > 0$, the function $\boldsymbol{w}(t,x)$ satisfies system (1) in the usual sense.

Let $(t_0, x_0) \in E$ and $u(t_0, x_0) > 0$. Because the sequence $\boldsymbol{w}_n(t,x)$ converges uniformly to $\boldsymbol{w}(t,x)$, it is possible to indicate a number N such that, for all $n \geqslant N$, the inequalities $u_n(t,x) > 0$ are satisfied in a certain rectangle Π_0 containing the point (t_0, x_0). Then, the sequence $\{\boldsymbol{w}_n(t,x)\}$ $(n \geqslant N)$ is compact in $C^{2+\alpha}(\Pi_0)$, and, consequently, the limiting function $\boldsymbol{w}(t,x)$ has derivatives $\boldsymbol{w}_t$, $\boldsymbol{w}_x$, and $\boldsymbol{w}_{xx}$ in the rectangle Π_0 and satisfies system (1) in the usual sense. In particular, Theorem 7.1 in [3] implies that the function $\boldsymbol{w}(t,x) \in C^{2+\alpha}(\Pi_0)$ and is a classical solution of system (1) in the domain Π_0. In addition, lemma in [7] implies that $\lim\limits_{x\to\infty} \boldsymbol{w}(t,x) = \boldsymbol{m}_\infty$. The theorem is proved.

REFERENCES

1. N. V. Khusnutdinova, “Thermal boundary layer on a plate,” *Dokl. Akad. Nauk SSSR*, **285**, No. 3, 605–608 (1985).
2. S. A. Kalashnikov, “On the occurrence of singularities in solutions of the equations of unsteady filtration,” *Zh. Vychisl. Mat. Mat. Fiz.*, **7**, 440–444 (1967).
3. O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Uraltseva, *Linear and Quasilinear Equations of the Parabolic Type* [in Russian], Nauka, Moscow (1967).
4. N. V. Khusnutdinova, “Boundedness conditions for the gradient of solutions of degenerate parabolic equations,” in: *Dynamics of Continuous Media* (collected scientific papers) [in Russian], No. 72, Inst. of Hydrodynamics, Sib. Div., Acad. of Sci. of the USSR, Novosibirsk (1985), pp. 120–123.
5. S. N. Kruzhkov, “Quasilinear parabolic equations and systems with two independent variables,” in: *Proc. Petrovskii Seminar* [in Russian], No. 6, Izd. Mosk. Gos. Univ., Moscow (1979), p. 217.
6. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics* [in Russian], Izd. Leningr. Gos. Univ., Leningrad (1950).
7. N. V. Khusnutdinova, “Conditions of global solvability of boundary-value problems for the steady-state boundary-layer equations for a compressible fluid,” in: *Dynamics of Continuous Media* (collected scientific papers) [in Russian], No. 75, Inst. of Hydrodynamics, Sib. Div., Acad. of Sci. of the USSR, Novosibirsk (1982), pp. 113–130.